Discussion on Type-I fuzzy boundary and Research on Boundary Definition of High Order Fuzzy Region

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Abstract
The definition of fuzzy boundary is a topic of great importance in research of modeling and analysis of fuzzy geographical phenomena. The problem "boundary syndrome" has been a longstanding problem in this domain, and this problem has seriously affected the research and application of fuzzy geographical model. The existing fuzzy boundary models were discussed at first, and then some models based on type-I fuzzy sets were analyzed in detail. This paper pointed out the fuzzy boundary models should have three kinds of meaning: "frontier", "transition" and "division". Three types of boundary models of high order fuzzy region were proposed based on interval type-II fuzzy set, and they embody three kinds of meaning of fuzzy boundary respectively. The models proposed by this paper have a positive effect on high order geographical phenomena modeling and analysis.

Keywords: fuzzy boundary, interval type-II fuzzy sets, high order vagueness

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1. Introduction
The definition of fuzzy boundary of fuzzy region is crucial and difficult in fuzzy geographical information science and it has direct effect on research of structure of fuzzy region, fuzzy topological relation, and direction relation and so on. There are lots of works on this domain, and existing products can be classified to two types. The first type is based on "three-valued logic", and the fuzzy boundary is usually named as "broad boundary". "Egg-Yolk" model is a classical model of this type [1], and the yolk around the core represents the uncertainty area of a fuzzy region in this model. This model is an extent of RCC model, and it does not allow the core or boundary empty. So this model could not represent the crisp object. Clementini and Di Felice proposed a simple broad boundary fuzzy object model based on point set theory which is similar with "Egg-Yolk" model [2]. Erwig and Schneider defined a fuzzy region model based on "three-valued logic" theory, a vague region is a pair of disjoint regions, and the first region is kernel and the second is boundary in this model [3]. Schmitz and Morris defined the fuzzy region through \( \alpha \)-cut set clusters [4], but this definition assumes that the boundary is broad everywhere and an \( \alpha \)-cut surrounds uniformly the core. Bejaoui et al. pointed out that this hypothesis is little realistic because regions can have a broad boundary in some locations and a sharp boundary in some others, and proposed an idea that shape vagueness contain three levels of shape vagueness: crispness, partial shape vagueness and complete shape vagueness, then they defined a new board boundary object model [5].

The second type definition to define the fuzzy boundary is based on classical fuzzy set theory (type-I fuzzy set). All of the first type definitions which had been discussed before ignored the possible change of attribute in boundary, so those definitions are not fine to describe the complex phenomena. Some researchers chose fuzzy logic theory to model the fuzzy spatial phenomena. Those type definitions can be divided into two subtypes. The first subtype definitions were proposed by mathematician and based on type-I fuzzy set completely. Three kinds of fuzzy boundary were defined respectively by Warren, Pu and Liu, Cuchillo-Ibanez and Tarres [6-8]. Atha and Ahmad extent Pu and Liu’s boundary as fuzzy semiboundary [9], and Dibyajyoti and Debajit extent other two fuzzy boundary as w-semi-boundary and c-semi-
boundary respectively [10]. Those definitions had rigorous mathematical foundation, and emphasized the fuzzy boundary of a fuzzy set. The second subtype definitions used fuzzy set theory combined with geographical information applications to define the fuzzy boundary of fuzzy geographical entities. Zhan proposed a simple fuzzy region model which is consists of three parts, and they are core, indeterminate boundary and exterior [11]. The indeterminate boundary is defined by $\alpha$-cuts, then it are divided to inside edge $A^\alpha$ and outside edge $A^{\bar{\alpha}}$. Tang gave a comparative analysis of three kinds of boundary defined respectively by Warren, Pu and Liu, Cuchillo-Ibanez and Tarres, and used Warren's definition to divide the structure of fuzzy region [12]. Liu and Shi defined a computational fuzzy topology to compute the interior, boundary and exterior of spatial objects. The computation is based on two operators, the interior operator and the closure operator [13], [14]. Bjørke defined a fuzzy boundary which is similar with Warren's definition [15].

It can be summarized that there exists a serious problem named as “boundary syndrome” in fuzzy geographical object modeling from the preceding analysis, and this problem restricted the development of modeling and analysis of fuzzy geographical entities. On the other hand, existing research focused on the type-I fuzzy boundary, and there is little discussion on high order fuzzy object’s boundary. The high order vagueness widely exist in the real world [16-19].and the high order fuzzy region model is more suitable to describe the complex geographical phenomena [20], nowadays research on high order fuzzy geographical phenomena is based on interval type-II fuzzy set. In fact, it must define fuzzy boundary before analyzing the topologic relation, distance relation and detail direction relation between high order fuzzy geographical objects [20-21].

The remainder of the paper is organized as follows. In Section 2, we present previous works on the boundary definitions based on theory type-I fuzzy set. Three kinds of type-I fuzzy boundary definition were discussed and proposed three kinds of meaning of fuzzy boundary in this section. In Section 3, we present three kinds of interval type-II fuzzy boundary to solve those problems discussed in this section. Finally, Section 4 presents our conclusions and discusses future research.

2. Boundary of type-1 fuzzy geographical object

Fuzzy boundary has three kinds of meaning. The first kind emphasizes the frontier of fuzzy object. The second emphasizes transition ambiguity between fuzzy entity and the external region. The third emphasizes dividing ambiguity between fuzzy entity and other entities (external). So we can induce three definitions of fuzzy boundary:

(1) The fuzzy boundary is the frontier region of fuzzy spatial object.
(2) The fuzzy boundary is the transition ambiguity between fuzzy spatial object and other entities (external).
(3) The fuzzy boundary is the dividing ambiguity between fuzzy spatial object and other entities (external).

This paper will not discuss the first type definitions discussed in section I, because they ignores the gradual change of the fuzzy boundary.In this section second type definitions would be reviewed and analysed.

2.1 Fuzzy boundary definition of type-I fuzzy geographical object

**Definition 1** [6] The fuzzy boundary of $A$ is the infimum of all closed fuzzy sets $D$ in $X$ with the property $D(x) \geq \overline{A}(x)$ for all $x \in X$ for which $\left( \overline{A} \wedge \overline{A^c} \right)(x) > 0$ or $A^c(x) \neq 1$.

**Definition 2** [7] The fuzzy boundary of a fuzzy set $A$ is the intersection of the closure with the closure of the complement of a fuzzy set.

**Definition 3** [8] The fuzzy boundary of $A$ is the infimum of all closed fuzzy sets $D$ in $X$ with the property $D(x) \geq \overline{A}(x)$ for all $x \in X$ for which $\left( \overline{A} - A^c \right)(x) > 0$.

$A(x)$ is a fuzzy set in a fuzzy topological space $(X, T)$. $\overline{A}(x)$ is the closure, $A^c(x)$ is the complement and $A^o(x)$ is the interior in above definitions.
Definition 4 [13-14] Let $A$ be a fuzzy set in $[0,1] = I^\circ$. For any fixed $\alpha \in [0,1]$, define the interior and closure operators on $A$ as $A = A_\alpha$ in $I^\circ$ and $A = A^\alpha$ in $I^\circ$ respectively. Where the fuzzy sets $A_\alpha(x)$ and $A^\alpha(x)$ in $X$ are defined by

$$
A_\alpha(x) = \begin{cases} A(x), & A(x) > \alpha \\ 0, & A(x) \leq \alpha \end{cases}
$$

$$
A^\alpha(x) = \begin{cases} 1, & A(x) > \alpha \\ A(x), & A(x) \leq \alpha \end{cases}
$$

Definition 5 [13-14] For $0 < \alpha \leq 1$, define the boundary of a fuzzy set $A$ in a fuzzy topology $(X, \tau_\alpha, \tau^{1-\alpha})$ as:

$$
(\partial A)_\nu(x) = A^{1-\alpha}(x) \land (A_\alpha)\nu(x)
$$

Definition 6 [15] Assume a fuzzy region $A$. The membership function of the fuzzy boundary $(\partial A)_\nu(x)$ of $A$ is defined as

$$
(\partial A)_\nu(x) = 2 \times A(x) \land A^\nu(x)
$$

2.2 Comparative analysis of fuzzy boundary definitions of type-I fuzzy geographical object

Some researchers have discussed the relationship between $(\partial A)_I$, $(\partial A)_II$, and $(\partial A)_III$ [9-10], [12]. Fuzzy membership grade of fuzzy boundary $(\partial A)_I$ would be reducing from the interior to the outside if the fuzzy region is a simple fuzzy region. It means that the degree of a point on space belonging to the region becomes weaker as the membership value changes smaller. Tang(2004) thought that fuzzy boundary $(\partial A)_I$ is more suitable for the geographic information applications, and the space extent of this boundary is the region while $\big(\overline{A} \cap A^\nu\big)(x) > 0$. Many studies used fuzzy boundary $(\partial A)_II$ when discussing fuzzy topological problems, and its membership first increases and then decreases from the interior of the fuzzy region to the outside. So this boundary focuses on a gradual transition of the fuzzy region to the outside. Fuzzy boundary $(\partial A)_III$ requires satisfying the constraint $\overline{A}(x) > A^\nu(x)$ in fact only several ring could meet this constraint for a simple 2d fuzzy set. So maybe this boundary definition is not suit for geographic information applications (Tang (2004) gave a full discussion on this problem).

Fuzzy boundary $(\partial A)_IV$ is a calculation model which is induced by $\alpha$ -cuts, and its extent is the space while $\big(\overline{A} \cap A^\nu\big)(x) > 0$. The membership function of fuzzy boundary would be change if $\alpha$ changes, and its value would be constant equal to 1 when $\alpha$ equals to 1. There is an intersection between the fuzzy boundary and interior of the fuzzy region under normal case. The space extent of interior becomes smaller while the value of $\alpha$ increases. The space extent of fuzzy boundary is stationary, but the membership grade of fuzzy boundary maybe change. The model proposed by Liu and Shi [13-14] maybe exist a overlap region between interior and boundary, and the extent of overlap region is determined by $\alpha$ . The fuzzy boundary definition $(\partial A)_IV$ is not strict; because $(\partial A)_IV$ maybe an open fuzzy set if $A(x)$ is open, so it can be redefined as:

$$
(\partial A)_IV(x) = 2 \times \overline{A}(x) \land \overline{A^\nu}(x)
$$

Boundary $(\partial A)_IV(x)$ has the same range with $(\partial A)_I(x)$ and $(\partial A)_IV(x)$. 

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It could be included that these boundary definitions have same spatial range in domain \(X\), and it has \(\bar{A} \cap \overline{A^c}(x) > 0\) in that range, but the membership function (grade) of these kinds of boundary are different. The other same point is that the trends of membership grade from interior to outside be similar under these boundary definitions \((\partial A)_m, (\partial A)_v\) and \((\partial A)_b(x)\).

Boundary \((\partial A)_m(x)\) has a ring in which the membership grade constant equal to \(1\), and boundary \((\partial A)_v\) maybe have a ring or belt and the width of this belt is determined by \(\alpha\). So \((\partial A)_m(x)\) emphasize “frontier”, and \((\partial A)_v(x)\) emphasize “transition”, and \((\partial A)_b(x)\) and \((\partial A)_b(x)\) emphasize “division”.

The “boundary syndrome” has seriously affected the modeling and analysis of fuzzy geographical phenomenon, but many definitions of fuzzy region are purpose-oriented, and people can choose different membership functions for defining fuzzy region in different application. The applications of fuzzy boundary definitions are purpose-oriented too. So we propose a formal fuzzy boundary definition and three kinds of specific fuzzy boundary definitions corresponding to kinds of boundary meaning in next section. Which fuzzy boundary definition should be chosen is determined by the application, so the “boundary syndrome” problem can be reasonably solved.

3 The fuzzy boundary of type-II fuzzy geographical object

The fuzzy boundary definition is a precondition of topological analysis among fuzzy regions. It assumed that the type-I fuzzy region is normal convex and upper semi-continuous in fuzzy 9-intersection Model, so this assumption would be adapted in this paper. The interval type-II fuzzy region model proposed by Guo and Cui [20] would be adapted and the indeterminate region of this model is regarded as the fuzzy boundary region, it means that lower or upper membership grade is lower than \(1\) and greater than \(0\).

3.1 The fuzzy boundary definition of type-II fuzzy geographical object

Three kinds of interval type-II fuzzy boundary are defined corresponding to three kinds of boundary meaning: “frontier”, “transaction” and “division”, showed as Figure 1.

![Figure 1. boundary of interval type-II fuzzy set. (a) Interval type-II fuzzy set. (b) Fuzzy boundary definition \((\partial \tilde{A})_m\). (c) Fuzzy boundary definition \((\partial \tilde{A})_v\). (d) Fuzzy boundary definition \((\partial \tilde{A})_b\).](image)

**Definition 7** The fuzzy boundary of an interval type-II fuzzy set \(\tilde{A}\) is the intersection of the closure with the closure of the complement of an interval type-II fuzzy set. This definition is extended from type-I fuzzy boundary definition \((\partial A)_b\) and it can be formally expressed as:
\[
\left(\partial \tilde{A}\right)_I(x) = \overline{A}(x) \land \overline{A}^c(x)
\]  

**Definition 8** The fuzzy boundary of an interval type-II fuzzy set \( \tilde{A} \) is the infimum of all closed fuzzy sets \( D \) in \( X \) with the property \( D(x) \geq \overline{A}(x) \) for all \( x \in X \) for which \( \overline{A} \land \overline{A}^c \geq \overline{A}(x) \) for all \( \overline{A}^o(x) \neq 1 \). This definition is extended from type-I fuzzy boundary definition \( \partial A \) and it can be formally expressed as:

\[
\left(\partial \tilde{A}\right)_I(x) = \overline{A}(x), \overline{A} \land \overline{A}^c \geq \overline{A}(x) > 0
\]

**Definition 9** Let the height of fuzzy boundary of an interval type-II fuzzy set \( \tilde{A} \) is \( \tilde{h} = [\tilde{h}, \tilde{h}] \), then the fuzzy boundary of \( \tilde{A} \) can be formally expressed as:

\[
\left(\partial \tilde{A}\right)_III(x) = \frac{1}{\tilde{h}} \overline{A}(x) \land \overline{A}^c(x)
\]

![Figure 2. The interval type-II fuzzy region](image)

![Figure 3. The outer and three types' boundary of interval type-II fuzzy region.](image)
3.2 Properties of interval type-II fuzzy boundary definitions

Interval type-II fuzzy boundary definitions \( \tilde{\partial A} \), \( \hat{\partial A} \) and \( \tilde{\partial A} \) have some properties obviously:

1. Three kinds of interval type-II fuzzy boundary are fuzzy closure sets.
2. It has \( \tilde{x} \geq 0 \) in those kinds of fuzzy boundary range.
3. \( \tilde{\partial A} \) emphasize “frontier”, and \( \hat{\partial A} \) emphasize “transition”, and \( \tilde{\partial A} \) emphasize “division”.

3.3 Case study

Let \( \tilde{A} \) be an interval type-II fuzzy region, the lower membership function \( \mu(x, y) \) and upper membership function \( \overline{\mu}(x, y) \) are expressed as formula (9) and (10).

\[
\begin{align*}
\mu(x, y) &= \begin{cases} 1, & d \leq 9 \\ 1.5 - \frac{x^2 + y^2}{18}, & 9 < d \leq 27 \\ 0, & d > 27 \end{cases} \\
\overline{\mu}(x, y) &= \begin{cases} 1, & d \leq 16 \\ 1.8 - \frac{x^2 + y^2}{20}, & 16 < d \leq 36 \\ 0, & d > 36 \end{cases}
\end{align*}
\]

Where \( d \) is the distance from any point to center \((0, 0)\). This interval type-II fuzzy region is showed as Figure 2, and the exterior, fuzzy boundary \( \tilde{\partial A} \), \( \hat{\partial A} \) and \( \tilde{\partial A} \) of \( \tilde{A} \) are showed as Figure 3.

4. Conclusion

The existing harvests on type-I fuzzy boundary has been analyzed at first in this paper, and we proposed that the boundary should have three kinds of meaning: “frontier”, “transaction” and “division”. But the existing boundary definitions only have one of these kinds of meaning, and this situation has led the “boundary syndrome”. This paper developed three kinds of boundary definitions based on interval type-II fuzzy region proposed by Guo and Cui [20] and can express three kinds of boundary meaning. We will focus on the research of topological analysis, direction analysis, distance analysis and so on based on the result of this paper in the future work. So the result of this paper would actively promote the research on topological relation, direction relation and distance relation between high order fuzzy regions. Type-I fuzzy theory has successfully used in fuzzy controller domain and so on [21-22], but some limitations of it has discussed by some people [23], and designed new fuzzy controllers based on interval type-II fuzzy logical. Some complex phenomena can not be analyzed adequately and objectively, maybe those problems can be resolved by the interval type-II fuzzy theory, and there has lots of works to do.

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