Mechanical Theorem Proving in Geometry

Gao Jun-yu*, Zhang Cheng-dong
Cangzhou Normal University
*corresponding author, e-mail: gaojunyu8818@126.com

Abstract
Mechanical theorem proving in geometry plays an important role in the research of automated reasoning. In this paper, we introduce three kinds of computerized methods for geometrical theorem proving: the first is Wu’s method in the international community, the second is elimination point method and the third is lower dimension method.

Key words: geometric theorem, Wu’s method, elimination point method, lower dimension method

Copyright © 2012 Universitas Ahmad Dahlan. All rights reserved.

1. Introduction
Mechanical Proving, that is, mechanization method, is to find a method which can be computed steeply step according to a certain rules. Today we usually refer it as "algorithm". The algorithm is applied in the computer programming, mathematical mechanization, and mathematical theorems. This realizes mathematical theorem to be proved with computer.

Mathematics in ancient China is nearly a kind of mechanic mathematics. Today, the method of Cartesian coordination gives this direction a solid step, and provides a simple and clear method for the proof of geometric theorems mechanization.

The mechanical thought of mathematics in ancient China made a deeply influence in Wu Wen-jun’s work about mathematics mechanization. In 1976, academician Wu Wen-jun began to enter the field of mathematics mechanization. Since then, he forwards to the mechanization method which establish theoretical basis for the mechanization of mental work. He proves a large class of elementary geometry problems by computer. It is unprecedented in our country. This is a machine proving method and known as Wu’s method throughout the world. It is the first system for mechanic proving method and it can give the proof for nontrivial theorem. He makes the Study of Theorem Proving in Geometry more mature.

In 1992, Academician Zhang Jing-zhong visited the USA to research mathematics mechanization and cooperated with Zhou Xian-qing and Gao Xiaos-han. The elimination point method is one that is based on area method. This brings readable Proof to be realized by computer for the first time. This result is significant for academic and mechanical theorem proving in geometry.

In 1998, Yang Lu created lower dimension method. He obtains the achievement in the mechanic proof of inequalities. The achievement of this method is as good as Wu’s method and elimination point method. It is a great achievement in the field of mechanical theorem proving in geometry by Chinese mathematicians.

Next, we will introduce the three methods respectly.

2. Wu’s method
Wu Wen-jun presents a method which is called Wu’s method. It is based on Quaternion of traditional mathematics. This method has been solves a series of actual problems in theoretical physics, computer science and other basic science fields. We can use Wu’s method to find a proof for the geometric theorem in computer. We introduce three main steps of this method as follows:
Step 1 Choosing a good coordinate system, free variable and restrict variable.
Let us denote the free variables as $u_1, u_2, \cdots, u_n$, and suppose they have nothing to do with the conditions of geometric problems. Similarly, let us denote the restrict variables...
as \( x_1, x_2, \cdots, x_m \) which are restricted by the conditions of geometric problems. In this way, a geometric problems turns into a polynomial problem:

\[
\begin{align*}
  f_1(u_1, u_2, \cdots, u_n, x_1, x_2, \cdots, x_m) &= 0 \\
  f_2(u_1, u_2, \cdots, u_n, x_1, x_2, \cdots, x_m) &= 0 \\
  \cdots \\
  f_m(u_1, u_2, \cdots, u_n, x_1, x_2, \cdots, x_m) &= 0
\end{align*}
\]

(1)

The conclusions of geometric problem can be expressed as a polynomial problem.

\[
g(u_1, u_2, \cdots, u_n, x_1, x_2, \cdots, x_m) = 0
\]

(2)

Or, it can be represented as a family of polynomials.

Step 2 Triangulation

According to the restrict variable, the rearrange of (1) is referred as triangulation. In another word, the systems of equation (1) is changed as:

\[
\begin{align*}
  f^*_1(u_1, u_2, \cdots, u_n, x_1^*) &= 0 \\
  f^*_2(u_1, u_2, \cdots, u_n, x_1^*, x_2^*) &= 0 \\
  f^*_3(u_1, u_2, \cdots, u_n, x_1^*, x_2^*, x_3^*) &= 0 \\
  \cdots \\
  f^*_m(u_1, u_2, \cdots, u_n, x_1^*, x_2^*, x_3^*, \cdots x_m^*) &= 0
\end{align*}
\]

(3)

Step 3 Gradual Division

Denote the polynomial \( f_j^* \) in (3) as \( f_j^* \) \( (j = 1, 2, \cdots, m) \), \( g \) in (2) is divided by \( f_m^* \), and the remainder of division algorithm is denoted as \( R_m \).

In order to avoid the fractional in quotient, we multiply \( c_1 \) to \( g \), that is,

\[
  c_1 g = a_1 f_m^* + R_m
\]

(4)

The remainder \( R_m \) is divided by \( f_{m-1}^* \)

\[
  c_2 R_m = a_2 f_{m-1}^* + R_{m-1}
\]

(5)

So, repeating this division, at last we get:

\[
  c_m R_2 = a_m f_1^* + R_1 (\equiv R)
\]

(6)

Then, let us iterate all the equations above and replace the coefficient of \( f_i^* \) \( (i = 1, 2, \cdots, m) \) of all equations above with \( a_i \) \( (i = 1, 2, \cdots, m) \), moreover, we obtain:

\[
  c_1 c_2 \cdots c_m g = a_1 f_m^* + a_2 f_{m-1}^* + \cdots + a_m f_1^* + R_1 (\equiv R)
\]

Dividing two cases to discuss: with \( R = 0 \) and \( R \neq 0 \).
Case 1: If \( R = 0 \), then, under the condition of \( f_i = 0 (i = 1, 2, \cdots, m) \) and the non-degenerate condition \( c_i \neq 0 (i = 1, 2, \cdots, m) \), there is \( g = 0 \), the required result follow.

Case 2: If \( R \neq 0 \), then the proposition is not true.

We present an example to show how to use Wu’s Method in solving problem.

Example 1 The problem is: The midline on the hypotenuse in a right-angle triangle equals to half of the hypotenuse.

Using Wu’s Method to get the solution of the above problem is: as shown in Figure 1, first, choose coordinates to right-angle triangle. The two right-angle sides of \( AO \) and \( BO \) are as \( x \) axis and \( y \) axis respectively. The vertex of the right angle is \( O \). We take \( D \) as the midpoint of hypotenuse and set up their coordinates as \( O(0,0) \), \( A(2u_1,0) \), \( B(0,2u_2) \) and \( D(x_1,x_2) \) respectively.

![Figure 1. Using Wu's Method](image)

Because \( O,A,B \) are arbitrary, this indicate that coordinates of \( u_1,u_2 \) are free variables and the midpoint \( D \) is restricted by the hypothesis, so the coordinates of \( x_1,x_2 \) are restrict variables.

Applying Wu’s Method, we solve this problem as follows:

**Step 1** Choosing a good coordinate system, free variable and restrict variable.

Supposing \( D \) is the midpoint of the \( AB \), by the midpoint formula we can get:

\[
 f_1 = x_1 - u_1 = 0 \\
 f_2 = x_2 - u_2 = 0
\]

Using step 1 in Wu’s Method, we only need to prove that the \( OD = BD \), by the distant formula:

\[
 g = (x_1^2 + x_2^2) - (u_1^2 + u_2^2) \\
 = (x_1 + u_1)(x_1 - u_1) + (x_2 + u_2)(x_2 - u_2) = 0
\]

**Step 2** Triangulation

Because \( f_1 \) just has the restrict variable \( x_1 \), and \( f_2 \) just has the restrict variable \( x_2 \), so \( f_1,f_2 \) themselves have been trianglize. So, it can be written as:

\[
 f_1^* = f_1 = x_1 - u_1 = 0 \\
 f_2^* = f_2 = x_2 - u_2 = 0
\]
Step 3 Gradual Division

\[
g \text{ is divided by } f_2^*, \text{and the division is:}
\]

\[
\frac{g}{f_2^*} = \frac{(x_1 + u_1)(x_1 - u_1) + (x_2 + u_2)(x_2 - u_2)}{x_2 - u_2} = \frac{(x_2 + u_2) + (x_1 + u_1)(x_1 - u_1)}{x_2 - u_2}
\]

That is, where

\[
g = (x_2 + u_2)f_2^* + R_2
\]

(7)

\[
R_2 = (x_1 + u_1)(x_1 - u_1)
\]

(8)

Where \( R = 0 \).

Using (8) to express (7), we will receive

\[
g = (x_1 + u_1)f_1^* + (x_2 + u_2)f_2^* + R
\]

When \( R = 0 \), The proposition has been proved.

3. Zhang's Elimination Point Method

We introduce Zhang's Elimination Point Method as follows.

Zhang Jing-zhong gives an effective method for what is called elimination point method, the method is based on the ancient area method, mainly used for deleting the constraint points. This idea of Zhang's Elimination Point Method is relative to the assumed conditions and area method, and the order of vanish point depends on the final constraint points. Then it is eliminated from back to front one by one. At last, the left point is total eliminated, if the number is equal to the right number, the proposition is permitted.

![Figure 2. Four cases of public edge](image-url)
To use Zhang’s Elimination Point Method effectively, we repeat the public edge theorem in the following. Next, we give a commonly important theorem of elimination point method. Public edge theorem (1970, Zhang). If the line $PQ$ and line $AB$ to $M$, then $\frac{S_{PMB}}{S_{QAB}} = \frac{PM}{QM}$

Public edge have four cases are shown as (a)-(d) in Figure 2 respectively.

We give an example to express the Public edge theorem of Zhang’s theorem.

Example 2: The problem is : Verify that the diagonal of a parallelogram is mutual divided. To solve this problem is the following:
Put a parallelogram as a diagram in Figure 3.
1. Do a parallelogram $ABCD$.
2. Connecting the diagonals $AC, BD$ which intersect at $E$.

We only need to verify $AE = CE, BE = DE$. $E$ is the restriction point which finally made. So, firstly to remove the point $E$.

\[
\frac{AE}{CE} = \frac{S_{ABD}}{S_{CDB}} \quad \text{(Public edge theorem)}
\]

\[
= \frac{S_{ABE}}{S_{ABC}} = 1
\]

Figure 3. parallelogram

Therefore, we can prove $AE = CE$, and $BE = DE$ as the above similar way..

4. Lower Dimension Method
With the establishment of Wu’s method and elimination point method, the Machine Proving of automated theorem proving of equation theorem has been solved, but the Machine Proving of automated theorem proof of equation theorem has been difficult to achieve. Therefore, Yang Lu and many other scholars worked for the establishment of a new algorithm which we called lower dimension method. The work in the field of machine theorem proving by Chinese mathematicians is a milestone.

The lower dimension method can be divided into three courses:
(1) Work out about $x, y, z$ boundary surface of inequality $\Phi_0, \Phi_1, \cdots, \Phi_k$.
(2) Using the boundary surface of the first step the parameter space was divided into finite cell decompositions, we get many connected open sets: $D_1, D_2, \cdots, D_k$, then from the connected open sets we select checkpoint for at least one abbrevd $(x_r, y_r, z_r) \in Dr \ (r = 1, 2, \cdots, k)$.
(3) using the finite number of checkpoints $(x_1, y_1, z_1), (x_2, y_2, z_2), \cdots, (x_k, y_k, z_k)$ to verify the correctness of inequality. If every established the proposition is true, otherwise, the proposition is false.
5. Conclusion

The three methods present different ways to deal with different problems. All of them are important in automated reasoning fields, when someone discusses a problem in automated reasoning fields, the first he (or she) would consider which of the three methods is just best for the problem, and then, he (or she) will obtain the best consequences. We hope our introduction is good for him (or her), now and in the future.

References