Performance Comparison of EKF/UKF/CKF for the Tracking of Ballistic Target

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Abstract
The problem about state estimation of nonlinear system is very important in engineering, such as navigation, economics, tracking and so on. This paper studied three well-known nonlinear Kalman Filters for the tracking of a ballistic target. Extended Kalman Filter (EKF), Unscented Kalman Filter (UKF), and the Cubature Kalman Filter (CKF) are applied for estimating the position, velocity and the ballistic coefficient of the ballistic target. The theory formulation and computer simulation has been done for the comparison of the three nonlinear Kalman Filters. Results show that all of them can accomplish the estimation task, but the UKF and CKF both have higher accuracy and less computation cost than the EKF.

Keywords: Extended Kalman Filter, Unscented Kalman Filter, Cubature Kalman Filter, Tracking

1. Introduction
It is in fact that many of the problems to be faced in order to achieve full autonomy can be cast as estimation problems [1]. And most of the problems in real word are nonlinear. The most widely used nonlinear estimation is the Extended Kalman Filter (EKF), but in EKF, the nonlinear system and measurement models are simply linearized around the state currently estimated, and the usual Kalman Filter (KF) equations [2] are applied based on the resulting linear models. It is well known that due to the errors introduced by the linearization, the EKF is a sub-optimal and biased estimator, and the calculating of the Jacobian matrix is always a very difficult and error-prone process [3-7]. The Unscented Kalman Filter (UKF), which is first introduce by Julier and Uhlmann at the end of the last century, operate with the motivation that “it is easier to approximate a Gaussian distribution than it is to approximate an arbitrary nonlinear function or translation.” It propagates a carefully selected “sigma-points” through the intact nonlinear equations, and uses the results to capture the mean and covariance of the posterior distribution [5, 6]. Based on the spherical-radial cubature rule, Ienkaran Arasaratnam and Simon Haykin proposed a new nonlinear Kalman filter called as the Cubature Kalman Filter (CKF) in 2009, which improves the performance over the UKF [8]. The performance comparison of these nonlinear Kalman filters are accomplish in this paper, based on the problem of estimation about the state of a ballistic target.

The rest of this paper is organized as follow. The three nonlinear filters, EKF, UKF and CKF, are introduced in section 2. The tracking problem of a ballistic target is formulated in section 3. In section 4, numerical simulations of the three nonlinear filters are done based on the tracking problem. Finally some conclusions are drawn in section 5.

2. Nonlinear Filters
The nonlinear dynamic system can be expressed as below

\[
\begin{align*}
X_{k+1} &= f_k(X_k, W_k) \\
Z_{k+1} &= h_{k+1}(X_{k+1}, V_{k+1})
\end{align*}
\] (1)

Where \(X_{k+1}\) and \(Z_{k+1}\) denote the state vector with n-dimension and the measurement vector with m-dimension at step k+1 respectively. \(W_k\) and \(V_k\) denote the process noise with n-dimension and the measurement noise with m-dimension respectively, and satisfy...
\[
E[W^T W_j] = \delta_{ij} Q_k \\
E[V^T V_j] = \delta_{ij} R_k \quad \forall i, j \\
E[W^T V_j] = 0
\]

2.1. Extended Kalman Filter

Let \( \hat{X}_{k-1,k} \) be the state estimate at time \( k \), and let \( P_{k-1,k} \) be the estimation error covariance matrix at the same time. Then the EKF prediction and correction equations are as follows[9-11]:

**Prediction:**
\[
\hat{X}_{k+1,k} = f_k(\hat{X}_{k-1,k}, W_k) \\
P_{k+1,k} = F_k P_{k-1,k} F_k^T + G_k Q_k G_k^T
\]

**Correction:**
\[
\hat{X}_{k+1,k} = \hat{X}_{k+1,k-1} + K_{k+1} (Z_{k+1} - \hat{Z}_{k+1,k}) \\
P_{k+1,k} = (I - K_{k+1} H_{k+1}) P_{k+1,k} \\
K_{k+1} = P_{k+1,k} H_{k+1}^T (H_{k+1} P_{k+1,k} H_{k+1}^T + R_k)^{-1}
\]

Where \( F_k \) and \( H_{k+1} \) are the Jacobians of the system equation and the measurement equation as bellow:
\[
F_k = \left[ \frac{\partial f_k}{\partial X_k}(\hat{X}_{k-1,k}, W_k) \right] \\
G_k = \left[ \frac{\partial f_k}{\partial W_k}(\hat{X}_{k-1,k}, W_k) \right] \\
H_{k+1} = \left[ \frac{\partial h_{k+1}}{\partial X_k}(\hat{X}_{k+1,k}, V_{k+1}) \right]
\]

The EKF can achieve satisfactory results for many applications, but may suffer from large estimate errors when systems have strong nonlinearities, and also suffer from the computation burden of the Jacobians.

2.2. Unscented Kalman Filter

Motivated by the intuition that “it is easier to approximate a Gaussian distribution than it is to approximate an arbitrary nonlinear function or translation.” by carefully selecting the “sigma-points” to capture the mean and covariance of the posterior distribution, Julier and Uhlmann achieved the UKF algorithm. It needn’t linearize the system and measurement equations as required by the EKF.

Let n-dimension state vector \( X_k \) with mean \( \hat{X}_{k,k} \) and covariance \( P_{k,k} \) be approximated by 2n+1 weighed sigma points. Then one type of UKF is as bellow[3-6]:

**Sigma point generation:**
\[
\begin{align*}
\chi_{i,k}^0 &= \hat{X}_{k,k} \\
\chi_{i,k}^+ &= \hat{X}_{k,k} + \sqrt{(n + \lambda)P_{k,k}} \\
\chi_{i,k}^- &= \hat{X}_{k,k} - \sqrt{(n + \lambda)P_{k,k}} \\
&i = 1, 2 \cdots n
\end{align*}
\]
Where \((\sqrt{(n+\lambda)} P_{k,k})_i\) is the \(i\)th row or column of the matrix square root of \((n+\lambda) P_{k,k}\), where the Cholesky decomposition is needed.

**Prediction:**

\[
\chi_{k+1,k} = f_k(\chi_{k,k}, W_k)
\]

\[
\hat{X}_{k+1,k} = \sum_{i=0}^{2n} W_{i}^{(m)} \chi_{k+1,k}
\]

\[
P_{k+1,k} = \sum_{i=0}^{2n} W_{i}^{(c)} \left( \chi_{k+1,k} - \hat{X}_{k+1,k} \right)^T + Q_k
\]

Where \(W_{i}^{(m)}\) and \(W_{i}^{(c)}\) is the weight that is associated with the \(i\)th point, defined as:

\[
W_{0}^{(m)} = \frac{\lambda}{n+\lambda}
\]

\[
W_{0}^{(c)} = \frac{\lambda}{n+\lambda} + (1 - \alpha + \beta^2)
\]

\[
W_{i}^{(m)} = W_{i}^{(c)} = \frac{1}{2(n+\lambda)} \quad i = 1, 2, \ldots, 2n
\]

Where \(\lambda = \alpha^2(n + \kappa) - n\) is a scaling parameter with the constant parameter \(0 \leq \alpha \leq 1\), \(\beta\) and \(\kappa\) is the tuning parameters of UKF, which can be used to reduce the overall errors.

**Correction:**

\[
\hat{Y}_{k+1,k} = h_{k+1}(\chi_{k+1,k}, V_{k+1})
\]

\[
\hat{Z}_{k+1,k} = \sum_{i=0}^{2n} W_{i}^{(m)} \hat{Y}_{i}^{(m)}
\]

\[
\hat{X}_{k+1,k+1} = \hat{X}_{k+1,k} + K_{k+1}[Z_{k+1} - \hat{Z}_{k+1,k}]
\]

\[
P_{k+1,k+1} = P_{k+1,k} - K_{k+1} P_{ZZ,k+1,k+1} K_{k+1}^T
\]

\[
K_{k+1} = P_{ZZ,k+1,k} P_{Z,k+1,k+1}^T
\]

\[
P_{ZX,k+1,k} = \sum_{i=0}^{2n} W_{i}^{(c)} \left( \chi_{k+1,k} - \hat{X}_{k+1,k} \right)^T
\]

\[
P_{ZZ,k+1,k} = \sum_{i=0}^{2n} W_{i}^{(c)} \left( \hat{Y}_{i}^{(m)} - \hat{Z}_{k+1,k} \right)^T + R_k
\]

It is clearly that the Jacobians are not needed in the implementation of the UKF; another advantage of the UKF is that the UKF can estimate the mean and covariance of the state accurately the second order for any nonlinearity.

### 2.3. Cubature Kalman Filter

By reducing nonlinear filtering to a problem of how to compute integrals, Ienkaran Arasaratnam introduced a spherical-radial cubature rule based new nonlinear filter called Cubature Kalman Filter (CKF) in 2009 [8,12], where all the integrands are all of the form
Nonlinear function * Gaussian density

Then numerically compute the multi-dimensional weighted integral of the form

\[ I(f) = \int_{\mathbb{R}^n} f(X) \exp(-X^T X) dX \]  

By finding a set of cubature point \( \xi_i \) and weight \( \omega_i \) to approximate the integral as bellow

\[ I(f) \approx \sum_{i=1}^{m} \omega_i f(\xi_i) \]  

The cubature points can be achieved by a third-degree spherical-radial rule, where a total 2n points are employed when the dimension of the state is n. the cubature points can be achieved as:

\[ \xi_i = \sqrt{n} [1] \]  

\[ \omega_i = \frac{1}{2n} \quad i = 1, 2, \cdots, 2n \]  

Factorize the know covariance matrix \( P_{k,k} \)

\[ P_{k,k} = S_{k,k} S_{k,k}^T \]  

The cubature point can be computed as:

\[ \chi_{k,k} = S_{k,k} \xi_i + \hat{X}_{k,k} \quad i = 1, 2, \cdots, 2n \]  

Then the CKF algorithm can be summarized:

**Prediction:**

\[ \chi'_{k+1,k} = f_k (\chi_{k,k}, W_k) \]  

\[ \hat{X}_{k+1,k} = \frac{1}{2n} \sum_{i=0}^{2n} \chi'_{k+1,k} \]  

\[ P_{k+1,k} = \frac{1}{2n} \sum_{i=0}^{2n} \chi_{k+1,k}^T \chi_{k+1,k} - \hat{X}_{k+1,k} \hat{X}_{k+1,k}^T + Q_k \]  

**Correction:**

\[ P_{k+1,k} = S_{k+1,k} S_{k+1,k}^T \]  

\[ \chi'_{k+1,k} = S_{k+1,k} \xi_i + \hat{X}_{k+1,k} \quad i = 1, 2, \cdots, 2n \]  

\[ \hat{Z}_{k+1,k} = h_{k+1} (\chi'_{k+1,k}, V_{k+1}) \]  

\[ \hat{K}_{k+1,k} = \frac{1}{2n} \sum_{i=0}^{2n} \hat{Z}_{k+1,k} \]  

\[ \hat{X}_{k+1,k+1} = \hat{X}_{k+1,k} + K_{k+1} [Z_{k+1} - \hat{Z}_{k+1,k}] \]  

\[ P_{k+1,k+1} = P_{k+1,k} - K_{k+1} P_{Z,k+1,k} K_{k+1}^T \]  

Where

\[ K_{k+1} = P_{X,k+1,k} P_{Z,k+1,k}^{-1} \]  

\[ P_{X,k+1,k} = \frac{1}{2n} \sum_{i=0}^{2n} \chi_{k+1,k}^T \hat{Z}_{k+1,k} - \hat{X}_{k+1,k} \hat{Z}_{k+1,k}^T \]  

\[ P_{Z,k+1,k} = \frac{1}{2n} \sum_{i=0}^{2n} \hat{Z}_{k+1,k}^T \hat{Z}_{k+1,k} + R_k \]
We can see that the CKF is like the EKF and UKF in structure, especially the same as the UKF in the implementation by “points”, but there is no other parameter need to be tuning in the CKF.

There is another type of CKF which is called SCKF (Square-root cubature Kalman Filter), who essentially propagate square-root factors of the predictive and posterior error covariance, hence avoid matrix square-root operations, also improve the numerical accuracy.

3. Tracking of a Ballistic Target

The problem of tracking a ballistic target was studied by many researchers in the area of estimation, because it has significant nonlinearities in the process and the observation models and has been analyzed extensively in literatures[3, 9], the ballistic target reenters the earth’s atmosphere after having traveled a long distance, its speed is high and the remaining time to ground impacts is relatively short. The goal of the tracking radar is to intercept and track the ballistic target by measuring the range corrupted by Gaussian measurement noise. This tracking problem is highly difficult because the target’s dynamics change rapidly and has significant nonlinearities. The sketch map is shown in fig. 1. We wish to estimate the position \( x_1(t) \), velocity \( x_2(t) \), and constant ballistic coefficient \( x_3(t) \) of a target as it reenters the atmosphere at a very high altitude at a very high velocity. The equations for this system are:

\[
\begin{align*}
\dot{x}_1 &= x_2 + w_1 \\
\dot{x}_2 &= \rho_0 \exp(-x_1/k) x_2^2/(2x_3) - g + w_2 \\
\dot{x}_3 &= w_3 \\
y(t_k) &= \sqrt{M^2 + (x_1(t_k) - a)^2} + v_k
\end{align*}
\]

Where \( w_i \) is zero-mean, uncorrelated noise that affects the ith process equation, and \( v \) is the uncorrelated measurement noise. \( \rho_0 \) is the air density at sea level, \( k \) is a constant that defines the relationship between air density and altitude, and \( g \) is the acceleration due to gravity.

We will use the continuous-time system equations to simulate the system, and suppose that we obtain range measurements every 0.5 seconds. The constants that we will use are given as

\[
\begin{align*}
\rho_0 &= 2 \text{ lb - sec}^2/\text{ft}^4 \\
g &= 32.2 \text{ ft/sec}^2 \\
k &= 20,000 \text{ ft} \\
E[v_i^2] &= 10,000 \text{ ft}^2 \\
E[w_i^2(t)] &= 0 \quad i = 1, 2, 3 \\
M &= 100,000 \text{ ft} \\
a &= 100,000 \text{ ft} \\
\end{align*}
\]

The initial conditions of the system and the estimator are given as

\[
\begin{align*}
x_0 &= \begin{bmatrix} 300,000 & -20,000 & 0.001 \end{bmatrix}^T \\
\hat{x}_0^* &= x_0 \\
\tilde{P}_0^* &= \begin{bmatrix} 1,000,000 & 0 & 0 \\
0 & 4,000,000 & 0 \\
0 & 0 & 10 \end{bmatrix}
\end{align*}
\]
We use rectangular integration with a step size of 1 msec to simulate the system, the simulated true position and velocity of the ballistic target is shown in Fig.2. For the first few seconds, the velocity is constant. But then the air density increases and drag slows the falling object. Toward the end of the simulation, the object has reached a constant terminal velocity as the acceleration due to gravity is canceled by drag. After about 10s, the altitude of the target is the same as that of the radar and range information provides less data about target motion.

![Diagram of a ballistic target](image)

**Figure 1. Tracking of a ballistic target**

**Figure 2. The simulated true position and velocity of the ballistic target**

4. Simulation of the Performance Comparison

The EKF, UKF and the CKF are all applied for the estimation problem, the simulation times are 60 seconds. Like in ref[13], Matlab is utilised for the simulation. Fig.3-Fig.5. shows typical EKF, UKF and CKF estimation-error magnitudes for this system. It is seen from the position and velocity estimates both spike around 10seconds, at which point the altitude of the measuring device and the falling target are about the same, so the measurement gives less information about the target’s altitude and velocity. It is seen from the figure that the UKF consistently gives estimates that are one or two orders of magnitude better than the EKF. And the CKF performed better the UKF. The elapsed time of EKF, UKF and CKF are 0.416874, 0.262409 and 0.257230 seconds respectively.

![Graphs showing position and velocity error](image)

**Figure 3. Position estimation error of the ballistic target**

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5. Conclusion
The comparison of three different nonlinear Kalman Filters for the tracking of a ballistic target has been analyzed, from the theoretical analysis and simulation results, we can conclude that all of the three nonlinear Kalman filters can accomplish the estimation task, but the EKF has the worst accuracy, and biggest computation burden, while the CKF can achieve the best performance. The UKF and CKF are much more easier to implemented because no linearization steps is need, eliminating the derivation and evaluation of Jocobian matrices, so the simulation time is less then that of the EKF.

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