Inertia Force Identification of Cantilever under Moving-Mass by Inverse Method

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Abstract
In this paper, a recursive inverse method is applied to solve the identification problem of inertia force between the cantilever and moving mass. The recursive inverse method consists of two parts: Kalman filter and recursive least-square algorithm. The basic Euler-Bernoulli beam model is introduced. Then, the differential equations and the state space model of the modal responses and the inertia force can be obtained. Finally, the recursive inverse method, which is based on the discretized state function of the system, is adapted. The identification results show that the recursive inverse method is suitable to be adapted in this problem. Some characteristics of the identification results are discussed and some further conclusions are reached.

Keywords: Recursive inverse method, Kalman filter, Identification, Cantilever, Inertia force

1. Introduction
The inertia force identification between cantilever and moving mass is a very important problem in the civil and structure engineering fields. Some important characteristics of cantilever structure can be obtained by accessing to the inertia force between moving mass and cantilever. However, it is difficult to measure the inertia force between cantilever and moving mass directly because the mass is in motion and the inertia force is time-varying.

Therefore, a lot of classical methods for the inertia force identification between simply supported beam and moving objects are presented by former researchers. Some of these methods are based on the Euler-Bernoulli beam model. S.S. Law investigated the Frequency-Time Domain Method (FTDM) [1], which access the inertia force spectrums by adapting the least-square method and then obtain the inertia force by using inverse Fourier transformation, and the Time Domain Method (TDM) [2], which identify the inner force by using the modal superposition principle in time domain. T.H.T Chan presented the Interpretive Method I (IMI) [3] and Interpretive Method II (IMII) [4], which are used to identify the moving loads according to the modal analysis of bridge responses. Minzhuo Wang [5] has improved the IMII, making it suitable to identify the inertia force between cantilever structure and moving mass.

R.E. Kalman [6] proposed the Kalman filtering technique and it has been widely used in many scientific and technical fields: objects’ tracking [7], signal processing [8], identification [9], control [10], etc. Since the Kalman filtering has a strong ability of data estimation and signal processing with the interference of noise, there are many other methods based on it to solve some inverse estimation problems. P.C. Tuan [11] proposed an inverse method to solve on-line two-dimensional inverse heat conduction problems. This method, which consists two parts: the Kalman filter and an estimator, estimate the input heat of two-dimensional rectangular region. The Kalman filter is used to generate the residual innovation sequence and then the estimator calculates the input heats by a recursive least-squares algorithm. This method can estimate the input heat efficiently with the interference of the noise.

C.K. Ma [12] adapted P.C. Tuan’s inverse method mentioned above and applied it in estimating many different kinds of position-constant input forces of a beam structures. The results showed that this inverse method can also estimate the position-constant input force accurately with the interference of the noise.
In this paper, the recursive inverse method is adapted to identify the inertia force between cantilever and moving mass. Firstly, we build the differential equations of the modal responses and the inertia force according to the improved identification model of Minzhuo Wang. Secondly, the state-space model, based on which we can get the discretized state function, is transformed from the differential equations. Thirdly, the recursive inverse method is adapted to identify the inertia force. Finally, some important characteristics of the identification results are obtained from the recursive inverse method are discussed and some advanced conclusions have been obtained.

2. Problem Formulation
2.1. Differential Equations of Modal Responses and Inertia Force

The interaction between the cantilever and the moving mass is a very intricacy process in the real physical environment. In this paper, Euler-Bernoulli beam model is chosen because simplified models can provide a more efficient connection between the parameters and the beam response than the complex ones [13]. The model of cantilever beam is shown in Figure 1.

Then, the motion equation of an Euler-Bernoulli beam can be given as[14]:

$$\rho \frac{\partial^2 v(x,t)}{\partial t^2} + EI \frac{\partial^4 v(x,t)}{\partial x^4} = \delta(x-ct) f(t)$$  \hspace{1cm} (1)

where $EI$ is constant flexural stiffness of cantilever, $\rho$ is constant mass per unit length of cantilever, $v(x,t)$ is the beam deflection at point $x$ and time $t$, $\delta$ is the Dirac delta function, $f(t)$ is the time-varying inertia force, and damping of the mass’s motion is neglected.

Based on the modal superposition theory, the solution of Eq. (1) can be obtained as:

$$v = \sum_{i=1}^{\infty} \sin \frac{i \pi x}{L} q_i(t)$$  \hspace{1cm} (2)

where $q_i(t)$ are the $i$-th modal displacements of the cantilever.

After substituting Eq. (2) into Eq. (1), each side of Eq. (1) is multiplied by $\sin(i \pi x / L)$. Then, the equation is integrated with respect $x$ to between 0 and L. Using the properties of $\delta(t)$ and the boundary conditions of simply supported beam:

$$v(0,t) = 0, \quad \frac{\partial^2 v(x,t)}{\partial x^2} \bigg|_{x=0} = 0,$$

$$v(x,0) = 0, \quad \frac{\partial v(x,t)}{\partial t} \bigg|_{t=0} = 0,$$

the equations can be obtained as:

$$\ddot{q}_j(t) + \omega_j^2 q_j(t) = \frac{2F}{\rho L} \sin j \alpha, \quad j = 1, 2, \ldots$$  \hspace{1cm} (3)
where
\[\omega^2_{n,j} = \frac{j^4 \pi^4 EI}{L^4 \rho}, \quad f_n(t) = F(t)\sin\left(\frac{n\pi x}{L}\right)\]
are the \(n\)th modal frequency and modal force respectively.

In this paper, because of the differences of the boundary conditions between the cantilever and simply supported beam, some parameters are adjusted and Eq. (3) needs to be modified as below:

\[\ddot{q}_j(t) + \alpha^2_{n,j}q_j(t) = \frac{F}{\rho L} \sin j\alpha x, \quad j = 1, 2, \ldots\]

However, the \(\omega\) in Eq. (4) can't be obtained as any form of analytical solution and we can only get the approximate solution of it based on the frequency equation of the cantilever as given below[15]:

\[\cos \beta_i L \cosh \beta_i L = -1\]

where \(\beta_i\) is the nodes of the \(i\)th main vibration mode with the initial position of the cantilever.

By using the numerical method, the solution of Eq. (5) can be obtained. According to the results above, the modal function of cantilever can be given as:

\[\varphi(i,x) = \cosh\left(\frac{\beta_i x}{L}\right) - \frac{\beta_i x}{L} \sinh\left(\frac{\beta_i x}{L}\right) - \frac{\beta_i}{L} \sin\left(\frac{\beta_i x}{L}\right)\]

where \(x\) is the position of the sampling point on the cantilever.

By combining the Eq. (4) and Eq. (6), the differential equations of the modal responses and the inertia force can be obtained as:

\[
\begin{bmatrix}
\ddot{q}_{1,k} \\
\ddot{q}_{2,k} \\
\vdots \\
\ddot{q}_{n,k}
\end{bmatrix}
+ \begin{bmatrix}
\alpha_{1}q_{1,k} \\
\alpha_{2}q_{2,k} \\
\vdots \\
\alpha_{n}q_{n,k}
\end{bmatrix}
= \frac{1}{\rho L}
\begin{bmatrix}
\varphi(1,l(k)) \\
\varphi(2,l(k)) \\
\vdots \\
\varphi(n,l(k))
\end{bmatrix}
P_k
\]

where \(\ddot{q}_{i,k}\) and \(q_{i,k}\) are the accelerate and displacement of \(i\)th vibration modal of cantilever respectively, \(P_k\) is the inertia force between moving mass and cantilever, \(n\) is the order of modal, \(l(k)\) is the distance of the sampling point from the start point on the cantilever and all these data are the \(k\)th one of their own data sequences.

### 2.2. State Space Model of the System

Eq. (7) can also be expressed as:

\[\dot{q} + \omega q = \frac{1}{\rho L} \varphi p\]

where \(\omega\) is the \(n \times n\) matrix of the modal frequency, \(q\) and \(\varphi\) are the \(n \times 1\) vector of the acceleration and displacement of modal responses respectively, \(\varphi\) is the \(n \times 1\) matrix of modal function and \(p\) is the inertia force between cantilever and moving mass.

From Eq. (8), the continuous-time state equations and measurement equation can be obtained as:

\[\dot{X}(t) = AX(t) + Bp(t)\]
\[Z_l(t) = HX(t)\]
A = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -\omega & 0_{n\times n} \end{bmatrix}, \quad B = \frac{1}{\rho L} \begin{bmatrix} 0_{n\times 1} \\ \varphi \end{bmatrix}, \quad X(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}.

H = I_{2n \times 2n} \text{ is the measurement matrix and } Z(t) \text{ is the observation vector.}

### 2.3. Discretized State Function of the System

Eq. (9) can be discretized over time intervals of length \( \Delta t \) with process noise input as below:

\[
X(k + 1) = \phi X(k) + \Gamma [p(k) + w(k)],
\]

\[
X(k) = \begin{bmatrix} q(k) \\ \dot{q}(k) \end{bmatrix},
\]

\[
\phi = e^{A\Delta t},
\]

\[
\Gamma = \int_{k \Delta t}^{(k+1)\Delta t} e^{A(k+1)\Delta t - x} B d\tau
\]

where \( X(k) \) is the state vector, \( \phi \) is the state transition matrix, \( \Gamma \) is the input matrix, \( p(k) \) is the inertia force that is to be identified, \( w \) is the input noise which is assumed to be zero mean and white with variance \( E[w(k_1)w(k_2)] = Q\delta_{k_1k_2} \), where \( Q \) is the process noise covariance and \( \delta_{k_1k_2} \) is the Dirac delta function.

Also we need to consider the measurement noise, therefore Eq. (10) can be expressed as below:

\[
Z(k) = HX(k) + v(k)
\]

where the observation vector:

\[
Z(k) = [Z_1(k) Z_2(k) \cdots Z_{2n}(k)]^T,
\]

and the measurement noise vector:

\[
v(k) = [v_1(k) v_2(k) \cdots v_{2n}(k)]^T
\]

is assumed to be zero mean and white that the variance of \( v(k) \) is:

\[
E[v(k_1)v^T(k_2)] = R\delta_{k_1k_2} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{2n}^2 \end{bmatrix},
\]

where the elements \( \sigma_i \) is the standard deviation of measurement noise of \( v_i(k) \).

### 2.4. Recursive Inertia Force Estimation Approach

This recursive estimation approach has two parts: Kalman filter and recursive last-square algorithm.

The Kalman filter is given as:

\[
\overline{X}(k / k - 1) = \phi \overline{X}(k - 1 / k - 1) + \Gamma (k - 1),
\]

\[
P(k / k - 1) = \phi P(k - 1 / k - 1)\phi^T + \Gamma Q\Gamma^T,
\]

\[
S(k) = HP(k / k - 1)H^T + R.
\]
The recursive least-square algorithm is given as:

\[ K(k) = P(k/k-1)H^T S^{-1}(k), \]
\[ P(k/k) = [I - K(k)H]P(k/k-1), \]
\[ \bar{Z}(k) = Z(k) - H\bar{X}(k/k-1), \]
\[ X(k/k) = \bar{X}(k/k-1) + K(k)\bar{Z}(k). \]

The steps of the identification process of the inertia force between cantilever and moving mass can be illustrated as below:

(i) Measure the vibration of the cantilever, transform it into the displacement and speed of the modal responses and form the observation vector \( Z(k) \) of the cantilever.

(ii) Obtain the innovation covariance \( S(k) \), Kalman gain \( K(k) \), innovation \( \bar{Z}(k) \) by using the Kalman filter.

(iii) Calculate the inertia force \( p(k) \) by substitution of \( K(k), S(k), \) and \( \bar{Z}(k) \) into the recursive least-square algorithm.

The detailed derivation of this estimation algorithm above can be found in the appendix of Tuan's work.

3. Numerical Calculation and Results

The deflection data of ten sampling points are acquired by numerical methods. The parameters of the cantilever in our model are: \( L \), the span length, is 1.5m, \( EI \), the constant flexural stiffness, is \( 3.42 \times 10^5 \) Nm² and \( \rho \), the constant mass per unit length, is \( 7.8 \times 10^3 \) kg/m³.

The sampling frequency is 10KHz.

In order to evaluate the effects of the recursive inverse method in our model, an evaluate criterion called relative percentage error (RPE)[16], which is adapted to compare the results of the identification with the numerical result, is defined as:

\[ RPE = \frac{|p(k) - \bar{p}(k)|}{\bar{p}(k)} \times 100\% \]
In this study, the recursive inverse method is used to identify the inertia force of a cantilever beam under moving mass. The flow chart of the recursive inverse method is shown in Figure 2.

The parameters and initial conditions for the recursive inverse method are given as:

\[
\begin{align*}
\bar{X}(-1/-1) & = [0 0 \cdots 0]^T, \\
p(-1) & = 0, \\
M(-1) & = 0_{nx}. 
\end{align*}
\]

Since \( P(-1/-1) \) and \( P_b(-1) \) are unknown at the beginning, they can be set as very large numbers. Therefore some initial estimation results will be “ignored”. \( P(-1/-1) \) and \( P_b(-1) \) are given as below:

\[
\begin{align*}
P(-1/-1) & = diag[10^6], \\
P_b(-1) & = diag[10^9].
\end{align*}
\]

Filter process noise covariance \( Q = 10^{-4} \), and based on the prior experience of the motion sensors, \( R \) is given as:

\[
R = diag \left[ 10^{-3} \ 10^{-5} \ 10^{-6} \ 10^{-2} \ 10^{-4} \ 10^{-4} \ 10^{-5} \right]
\]

As mentioned above, different value of \( \gamma \) will have different influence to the identification results. If \( \gamma \) is close to 1, it may lose some fast updating ability; but, if \( \gamma \) is close to

\[
RPE = \frac{\sum |f_{true} - f_{idem}|}{\sum f_{true}} \times 100\%
\]
0, it may bring us more estimate errors. We need a compromise and choose the value of $\gamma$ as 0.6.

The recursive inverse method was adapted in three situations of different velocities of moving mass: $v=5$ m/s, 10 m/s and 20 m/s. The relative percentage errors of three identification results are listed in Table.1, and the identification results are shown in the Figure 3-5.
According to the identification results as it shown in Fig.4-6, some characteristic of this recursive inverse method adapting in the identification of inertia force between cantilever and moving mass can be obtained as below:

(i) There is a very short delay of the identification process at the beginning. It will cost the recursive inverse method a little time to reach the numerical result from start point. In the cases of this paper, the delay takes about $3.2 \times 10^{-3}$ s, $2.9 \times 10^{-3}$ s, and $2.8 \times 10^{-3}$ s, respectively.

(ii) After the estimate results reach the numerical result for the first time, it will have some jitters which will be decreased after some time and then enter the “steady period”.

(iii) There are still some little jitters in the “steady period”, but the identification results and numerical result are highly consistent and the jitters become increasingly weak.

4. Conclusions

In this paper, a recursive inverse method is adapted to solve the identification problem of the inertia force between cantilever and moving mass. The recursive inverse method is based on two different parts: Kalman filter and recursive least-square algorithm.

Although there is a very short delay and some little jitters exist at the beginning of each identification process, the recursive inverse method adapted in this paper is very appropriate to identify the inertia force between the cantilever and moving mass:

(i) The relative percentage error of each result is very small and the identification result is highly consistent with numerical result, especially after the short period of the jitters.
(ii) Because of the advantages of Kalman filtering technique, this recursive inverse method can also calculate the identification results very fast and it only needs a little memory to store the data for the calculation.

Future work will focus on the influence of the recursive inverse method’s parameters on the identification results and optimize the identification results by adjusting these parameters.

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Reference